Modal reasoning as simulation

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Abstract

Characterizing deductive reasoning in terms of simulations with concrete, positive elements helps explain the integration of reasoning with other aspects of cognition. It can also help account for how cognitive abilities such as deductive reasoning and natural language interpretation evolved from the sensory-motor abilities shared by human ancestors. However, an obvious challenge to such an account of reasoning is whether it can adequately deal with the abstract elements in human reasoning that do not have obvious analogues in non-logical representations, such as quantifiers and negation. Our previous work showed that statements about ordinary simulations have at least the expressive power of first-order logic. In this paper, we show that by using the notion of inhibition on the inheritance of elements between simulations, we can similarly account for the fundamental elements of modal logic.

1 Introduction

Characterizing human reasoning in terms of mental simulations with concrete, positive elements, as in [2, 9], has both theoretical and empirical merits. The grounding of reasoning in non-symbolic representations [10] becomes easier. Also, assuming that the ingredients of reasoning are basically the same as the semantics of natural language expressions, this means that the meanings of natural language expressions are tightly grounded in non-logical representations such as visual imagery. This is crucial since human reasoning and natural language interpretation are known to continually interact with other cognitive processes that deal with non-logical representations, such as visual information processing [17]. A simulation-based theory of reasoning is also preferable from the evolutionary and developmental viewpoint. That is, human reasoning in such a theory is tightly based on the sensory-motor functions of the brain [11] and thus, this theory can naturally explain how human reasoning has emerged out of the cognitive mechanisms shared by human ancestors.

Despite the above merits, the challenge to the simulation theory of reasoning is whether it can deal with the abstract concepts that appear in reasoning, such as quantifiers, negation and counterfactual propositions, which are not easy to represent in non-logical representations such as visual images. In this regard, some might argue that logical symbolic systems that are equipped with quantificational, negative and modal operators [20, 14] provide better theoretical tools for enabling human reasoning and natural language interpretation, despite their greater deviance from non-logical representations such as visual imagery.

Addressing the above challenge, [18] showed that a fairly natural set of mental simulation mechanisms can deal with the natural language constructions whose semantics are often taken to

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motivate the logical operators and connectives, $\forall, \exists, \neg, \wedge, \vee, \rightarrow$, of first-order logic (FOL). What was essential there was to formalize mental simulations only with concrete, positive elements (i.e., excluding logical operators such as quantifiers and the negative operator), so that we can maintain the tight correspondence between logical and non-logical representations.[1] Even though our simulation language is free of logical operators such as quantifiers, we showed that it can be just as expressive as FOL. In this paper, we extend this approach and show that an operator-free simulation language can be as expressive as a modal logic (ML) with basic modal operators, such as $\Box\phi$ ('it is necessary that ϕ') and $\Diamond\phi$ ('It is possible that ϕ'). Our primary focus is on the basic modal propositions in which $\Box$ and $\Diamond$ are interpreted in terms of logical necessity and possibility and hence we mostly ignore further complications that motivate more complex modal logics such as epistemic logic, deontic logic and tense logic cf. [16, 22, 4]. Also, we only provide a rough sketch of how simulations without $\Box$ and $\Diamond$ operators can characterize basic modal propositions. The precise syntax and semantics of our simulation language, as well as a formal comparison with standard modal logics, is left for another paper.

In section 2, we summarize our treatment of some of the propositional operators of non-modal FOL by way of mental simulations, in order to show in section 3 that we can use basically the same simulation mechanisms to characterize modal propositions. In contrast, in logical approaches, a non-modal logic (such as classical propositional logic, cf. [21]) which deals with non-modal propositions and a modal logic (such as ML with $\Box$ and $\Diamond$, cf. [1]) which can deal with counterfactuals and other intensional propositions are fundamentally different inference systems. Section 4 summarizes this paper and suggests future research directions.

2 Basics

This section provides the basics of our simulation mechanisms that we extend in our account of modal reasoning in the next section.

First, consider a concrete state of affairs in which there is a red box to the left of a black ball. This can be captured in our language in (1).

(1) In reality, $\{Square(s), Red(s), Circle(c), Black(c), LeftOf(c, s)\}$

The commas that enumerate propositions in the set, $\{\phi_1, ..., \phi_n\}$, are interpreted as AND. There is evidence [23] that human perception such as vision can individuate objects and attribute certain properties and relations to those objects. Correspondingly, 'non-logical' computational mechanisms (which are often used for modeling perceptual information processing) usually can attribute at least some properties and relations to objects, cf. [15] for computer vision. Thus, using the atomic predicate-argument structure in the form of $Pred(t1, ..., tn)$ in reasoning does not break the correspondence between logical and non-logical information processing mechanisms. Similarly, enumerating such atomic statements does not compromise the correspondence between logical and non-logical representations. The conjunction of any number of concrete statements as in (1) can be captured in one image and non-logical computations normally deal with analogous data-structures, cf. [7].

We call the set in (1) a **reality set**. As the term indicates, the reality set represents the person's knowledge of what holds in reality.[2] The content of the reality set continually grows,

[1]Computationally, this allows us to spontaneously switch between logical and non-logical computational mechanisms depending on the purpose, cf. [13]. This is especially useful for human-level AI, which needs both logical and non-logical computational mechanisms spontaneously interacting with one another. See [8] for the importance of such a 'hybrid' computational architecture.

[2]We use the term 'reality' as a notational expedient and what reality exactly means in cognition is not

accumulating the information that the person gains either via his own perception or from other information sources.

Not every proposition that appear in human reasoning can be captured in one image in the way with (1) above. For example, conditional, disjunctive, quantified and negative propositions cannot be easily represented in this way. Our previous work, [18], showed that a fairly natural set of simulation mechanisms can characterize these propositions. Here, we focus on our treatment of conditional and quantified propositions.

First, mental simulations usually involve some sort of 'assumption-conclusion' structure. For example, [6] assumes that the object representation possible in human mind is a mechanism of 'as-if' neural simulation. See also [11] for some neurological evidence that humans can simulate perceptual experiences in their mind, in a hypothetical setting that may involve assumptions. The latter work provides further physiological evidence that humans can simulate perception by activating the sensory areas of the brain so as to mimic the activity normally initiated by the sense organs.

In fact, some sort of assumption-conclusion structure seems to be involved in the actual processing of perceptual stimuli, such as visual computation. The famous pattern-completion task conducted by vision, cf. [3], can turn a dotted newspaper image into a complete image, and this process has the structure analogous to the assumption-conclusion structure.

(2) Dotted Image $\Rightarrow$ Complete Image

Visual information processing can 'elaborate' a dotted image into a complete image as in (2). We assume that this structure corresponds to the assumption-conclusion structure in simulations, as in (3).

(3) Simulation$_i$, based on: $\{DottedImage\}$ / Conclusion : $\{CompleteImage\}$

Given the above, consider a conditional statement, 'If that rock (= r) turns red, it means hot.' The object r here denotes a particular rock in reality. We let the reality set, 'In reality, $\{Rock(r)\}$,' represent this state of affairs. Based on this factual state of affairs, the agent (i.e., the person who reasons) can run a mental simulation, based on a hypothetical assumption that the object r turns red, and may conclude that r is hot, as in (4).[3]

(4) S1, based on: $\{TurnRed(r)\}$/ Con : $\{Hot(r)\}$

S1 in (4) together with the above reality set captures the conditional statement above.

S1 in (4) represents a simulation with a particular object that exists in reality. However, there is neurological evidence [11] that simulating with an imaginary object, for example, based on the mentally entertained hypothesis that an imaginary coin is tossed, is essentially the same as actually perceiving a coin tossed in reality, in terms of how the relevant areas of the brain for vision are activated. We take this as a justification for continually introducing 'new' objects into mental simulations, as in (5).

(5) S2, based on: $\{New(c), Coin(c), TossedInTheAir(c)\}$/ Con: $\{Fall(c)\}$

The syntactic sugar, '$New(c)$,' in the basis of S2 means that the object c is newly introduced

essential. Similarly, mentioning a person's knowledge of what holds in reality does not mean that we are concerned about epistemic logic. Our main focus is to provide formal tools that allow us to represent propositions without using operators such as $\Box$ and $\Diamond$ and what is essential here is only that the reality set is a monotonically growing set of formulas in the form of $Predicate(t_1, ..., t_n)$ and the formulas in this set are 'inherited' into simulations in the way that we explain below.

[3] We ignore tense, aspect and other complexity such as the internal structure of the predicate $TurnRed$.

there and therefore, the only properties that the agent attributes to c in (5) are the assumptions in the basis of S2, i.e., c is a coin tossed in the air.[4] Now, suppose that this simulation produces the conclusion that c falls with full certainty.[5] From this simulation result, we can tell that the agent believes that any object that is a coin and is tossed in the air falls, since otherwise, the agent would not have been able to conclude with full certainty[6] that c falls based only on the assumption that c is a coin tossed into the air. In this manner, from the result of the agent's mental simulation S2, we can definitely tell the presence of his universally quantified belief, 'Every coin tossed in the air falls,' or '$\forall x((Coin(x) \wedge TossedInTheAir(c)) \rightarrow Fall(c))$' in FOL. Since we can recover this proposition deterministically from the simulation result as above, in the formal simulation language, we can let S2 in (5) represent this universally quantified proposition.

S2 above does not imply that it is (logically) necessary that every coin tossed in the air falls and this subtlety becomes important in the next section. In our formulation, this is because everything that is true in reality is understood to be true in all factual (i.e., non-counterfactual) simulations. Technically, we say that what is true in reality is **inherited** by simulations. In our formulation, each simulation inherits all the formulas in the reality set by default (we discuss how to block this inheritance in the next section). Thus, a more precise implication that we can read off S2 in (5) is that according to the agent's current knowledge about real world, every tossed coin falls.

Next, the agent may run a mental simulation with an empty set of assumptions. As an intuitive example, suppose that the agent heard a gun-shot in reality. This gun-shot is then included as an atomic statement in the reality set. The gunshot does NOT count as an assumption of a simulation, since an assumption of a simulation is something that is assumed on top of the perceived facts in reality. However, the agent can still inherit this perceived reality of gunshot into a simulation with the empty basis set, as in (6), concluding that a bullet flying in the air will follow from that gunshot, according to his knowledge of the real world.

(6) S3, based on: $\{\}$/ Con: $\{New(b), Bullet(b), FlyingInTheAir(b)\}$

S3 means that responding to the gun-shot heard in reality, the agent introduced a new object b, which is a bullet flying in the air, in the conclusion of the simulation with no assumptions. Now, from this simulation result, we can definitely tell that there is a bullet flying in the air in the agent's mind, that is, '$\exists x(Bullet(x) \wedge FlyingInTheAir(x))$' in FOL. Since we can deterministically read this existential proposition off the result of the above simulation, in our formal simulation language, we can let S3 represent this existential proposition.

Technically, (5)$\sim$(6) means that an object that is newly introduced in the basis of a simulation implies universal quantification while an object that is introduced newly in the conclusion of a simulation implies existential quantification.

[4] In contrast, if an object is inherited from the reality set into a simulation, the properties attributed to the object in the reality set can also contribute to the simulation result.

[5] The term 'c' in the conclusion of S2 is 'old' and refers back to its original occurrence in the basis of S2.

[6] In this paper, we only consider 'fully certain' simulations, in which the relation '/' from the assumption to the conclusion corresponds to logical entailment (i.e., S2 in (5) means that (given the other contingent facts inherited from the reality set) '$Basis : \{Coin(c), TossedInTheAir(c)\}$' entails '$Con : \{Fall(c)\}$).' In order to accommodate uncertainty in human reasoning, we can add 'weights' to simulations so that, say, the basis is taken to lead to the conclusion to the degree suggested by the given weight. But we do not use uncertain simulations in this paper.

3 Modal propositions

In this section, we characterize modal reasoning using the same simulation mechanisms that we used for non-modal reasoning in section 2, only in a slightly different manner. First, we deal with a counferfactual[7] proposition in terms of a 'counterfactual' simulation. Consider (7).

(7) a. If Jack were a bird, he would fly.
b. $\Box(Bird(jack) \to Fly(jack))$
c. S5, based on: $\{^*Bird(jack), BeIn(jack, R)\}$/ Con: $\{Fly(jack)\}$

The simulation S5 in (7-c), which represents the proposition in (7-a) in our account, means that if Jack, who is a particular individual in reality, is counterfactually assumed to be a bird, it then follows that Jack flies. '*' in '$^*Bird(jack)$' indicates that this assumption contradicts the fact '$\neg Bird(jack)$' in reality.[8] This prevents S5 from inheriting the the formula '$\neg Bird(jack)$,' as well as some other propositions associated with this formula.[9]

'$BeIn(jack, R)$' in (7-c) is a 'syntactic sugar' that indicates that the individual *jack* is in the reality set, without explicitly showing the reality set. Since the term *jack* in S5 refers back to its original occurrence in the reality set, it does not imply universal quantification.

Note that the 'inheritance' mechanism above was already used in our account of non-modal reasoning in section 2. The only difference above is that unlike 'factual' simulations in section 2, which inherit all the formulas in the reality set, the simulation S5 in (7-c) does not inherit some formulas from the reality set. In this way, we can simulate what would happen if things were different from reality. As we discuss below, we can deal with logical necessity and possibility in a similar manner.

In comparison to S5 in (7-c), the modal logic formula in (7-b) is arguably too strong for the proposition in (7-a). First, its 'minimal logic' interpretation, i.e., 'In all logically possible worlds, $(Bird(jack) \to Fly(jack))$ is true', is too strong, since unlike a tautology such as '$2 + 2 = 4$', there is logical possibility that '$(Bird(jack) \to Fly(jack))$' is false (i.e., it is logically possible that '$Bird(jack)$' is true but '$Fly(jack)$' is false). We can use accessibility relations in the Kripke-frame interpretation models [12] in order to distinguish logical necessity such as '$\Box(2 + 2 = 4)$' (i.e., 'It is logically necessary that 2+2=4') from weaker necessities, say, '$\Box_e(Bird(jack) \to Fly(jack))$ (i.e.,'If Jack were a bird, he would fly').[10] The former proposition will be interpreted with the universal accessibility relation in which all the logically possible worlds are accessible from the world in which '$\Box(2 + 2 = 4)$' is evaluated. In contrast, we can evaluate '$\Box_e(Bird(jack) \to Fly(jack))$' with a more restrictive accessibility relation, so that, for example, only those possible worlds that the speaker of this utterance has in mind are accessible. It would then roughly mean that in all the *relevant* worlds in which Jack is a bird, he flies.[11]

[7] Although we use the term 'counterfactual' for convenience, what we mean with it is only that the proposition in question contradicts some fact in reality. Analogously, a 'counterfactual simulation' contradicts some known fact (and hence does not inherit the corresponding formula(s)), as we discuss below (7).

[8] We assume here that Jack is human in reality, which entails that he is not a bird in reality.

[9] We do not specify here exactly which propositions other than '$\neg Bird(jack)$' are NOT inherited by S5 because of the counterfactual assumption '$^*Bird(jack)$'. This decision will involve delicate semantic and pragmatic considerations. For example, although Jack is assumed to be a bird in S5, depending on the context, we might not want S5 to block all the human properties of Jack.

[10] Alternatively, modal logic could use possibility modality, say, '$\Diamond(Bird(jack) \land Fly(jack))$,' for (7-a). But this formula does not capture the conditional interpretation of (7-a). We could resolve this by using a more complex modal structure, but what is crucial here is that modal logic will need different kinds of necessity/possibility operators in order to be empirically adequate, which is reasonably uncontroversial in modal logic literature, cf [5].

[11] We use the word *relevant* here only for convenience.

Now, although this can solve the technical problem of distinguishing logical and contingent necessities, etc. with a standard modal logic, in order to be empirically adequate, this approach would need to consider the detailed semantics of the particular non-modal propositions such as '$2+2=4$' and '$(Bird(jack) \rightarrow Fly(jack))$', as well as some pragmatic factors, when it decides which modal operators to add to these propositions. This makes it difficult to define the syntax and semantics of the modal language including the forms such as '$\Box\phi$' and '$\Box_e\phi$' in the most general manner expected of a proper logical language, sound and complete to some well-defined semantic structures. But we do not discuss this issue any further in this paper. For further discussions about which logic-like systems are acceptable from a proof theoretic viewpoint, see [19].

In our account, logical necessity and weaker necessities can be distinguished in terms of which facts are inherited (or not inherited) from the reality set. For example, as we discussed with S5 in (7-c) above, other than the blockage of inheriting '$\neg Bird(jack)$' from reality (this blockage is required by the atom '$^*Bird(jack)$' in S5), what other facts are NOT inherited by S5 can depend on various factors, such as the context of the utterance and world knowledge that the agent has or considers relevant for this simulation. As a result, the simulation S5 in (7-c) does NOT imply that in every logically possible world in which Jack is a bird Jack flies. It only implies that in *some* possible worlds (i.e., in those possible worlds in which the facts are like the inherited facts in S5), if it is assumed that Jack is a bird, then it follows that Jack flies.

Deciding which propositions are inherited by each simulation poses a major technical challenge but the problem of inheritance is a general problem that needs to be solved in any computational mechanism that evaluates alternative states of affairs. Also, since our language does not include modal operators, we do not have the above-mentioned problem of defining the syntax and the semantics of the forms such as $\Box\phi$ and $\Box_e\phi$ in a sound and complete manner.

Ultimately, we will need to show that simulations can express all the essential propositions that standard modal logic can represent. But in his paper, we only sketch how we can handle the basic universal and existential modalities.

(8) a. S5, based on: $\{^*Bird(jack), BeIn(jack, R)\}$/ Con: $\{Fly(jack)\}$
b. S6, based on: $\{^*\}$/ Con: $\{BeIn(jack, R), BeRich(jack)\}$
c. S7, based on: $\emptyset \cup \{^*Bird(jack), BeIn(jack, R)\}$/ Con: $\{Fly(jack)\}$
d. S8, based on: $\emptyset$/ Con: $\{2+2=4\}$

We repeat S5 in (7-c) in (8-a). Again, the couterfactual assumption, $^*Bird(jack)$, tells us to block the inheritance of the fact '$\neg Bird(jack)$' in reality but other than that, we cannot deterministically tell which other facts are blocked from reality. Thus, S5 implies something similar to the possibility statement, 'In at least one counterfactual world in which John is a bird, John flies.'[12]

One may run a simulation that might not inherit all the facts from reality without specifying at all which facts are blocked. The syntactic sugar '*' inside the empty assumption set '$\{^*\}$' in (8-b) represents such a simulation. S6 means that if the agent simulates by assuming that something can be different from reality, then the simulation concludes that Jack is rich. In other words, it means that Jack could be rich if things were (or 'might be') different from reality, which corresponds to '$\Diamond Rich(jack)$' (i.e., 'Jack could be rich') in modal logic.

Alternatively, one may run a simulation only with the counterfactual assumption,'$^*Bird(jack)$', without inheriting anything from reality. (8-c) represents this simulation. '$\emptyset$' there means that

[12] Although this proposition basically corresponds to possibility modality in modal logic, we do not show a modal logic formula here because of the technical difficulty that we discussed in footnote 10. Note that unlike the modal logic formula in footnote 10, S5 has the conditional structure adequate for (7-a).

S7 inherits no formula from the reality set and hence the only thing that S7 depends on are the assumption,'$^*Bird(jack)$' (i.e., counterfactually, Jack is a bird in this simulation), and the fact, '$BeIn(jack, R)$' (i.e., '$jack$' denotes a particular individual in reality). Since the conclusion '$Fly(jack)$' is derived solely from these, S6 implies that if Jack were a bird, it follows that he would fly irrespective of other contingencies, corresponding to '$\Box(Bird(jack) \rightarrow Fly(jack))$' in modal logic. Finally, if we only put '$\emptyset$' in the basis of a simulation, as in (8-d), it means that this simulation is based on no assumptions at all and also that absolutely nothing is inherited from anywhere into this simulation. In other words, the conclusion '$2 + 2 = 4$' follows with absolutely no assumptions and without any reservations. As a result, this simulation implies that it is logically necessary that '$2 + 2 = 4$', or '$\Box(2 + 2 = 4)$' in modal logic.

We have shown that simulations can handle the basic possibility ('$\Diamond$') and necessity ('$\Box$') modalities. We leave our treatment of more complex modal propositions for another paper.

4 Conclusion

In this paper, we have shown that the basic elements of modal logic can be characterized in terms of mental simulations and the blocking of inheritance of elements between them. Our formal simulation language does not include modal operators such as '$\Diamond$' and '$\Diamond$' and thus, will help improve the efficient communication between the logical and non-logical computational mechanisms in a hybrid computation model of human-level intelligence. Also, since simulations of real world phenomena already imply that some things are different between reality and simulations and thus, the problem of which facts in reality are inherited from reality to simulations is a fundamental issue with the simulation theory of human reasoning, our approach is more explanatory than a traditional logical approach that adds fundamentally different elements (i.e., '$\Diamond$' and '$\Box$' together with the Kripke frame semantics) primarily for modal phenomena.

A thorough comparison of our simulation language with a standard modal logic language, as well as a first-order language that quantifies over world variables [1], is left for future research.

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