Senses, Sets and Representation

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Abstract

Frege and Russell agreed on propositions' structure but not on their contents. Instead of favoring one of these views over the other, we seek to reconcile both conceptions. We show that this can be achieved by demoting the Fregean Sense and generalizing the Cantorian set, in terms of a novel conception, termed "typed set". We argue that this is important to AI, as resorting to metaphysically murky, psychologically unappealing and computationally expensive notions in the semantic treatment of natural-language constructs is deemed unnecessary.

1 Do We Need Senses?

Russell's *On Denoting* introduces his now well-known theory of definite descriptions and a less known and mostly ignored argument—dubbed "Gray's Elegy"—against tripartite theories of meaning which admit the sign, what is designated by the sign and a third entity which mediates between the two. Russell's earlier theory of denoting concepts and Frege's theory of sense and reference fall into such category of theories, according to Russell. In the argument, Russell charges that theories which postulate entities—such as denoting concepts/complexes or senses—are doomed. Since, according to Russell, such theories are faced with the difficulties of: 1] expressing a proposition in which the alleged entities are the subject of the proposition; 2] demonstrating a logical relation—"denoting" as called by Russell—which must hold between the alleged entity and what it denotes, the referent. Further, Russell demands that this relation be genuinely logical and not just "linguistic through the phrase".

The literature is divided between those who share Russell concerns and those who do not, (see (Salmon, 2005) and references therein). However, we believe Russell's argument is too important to ignore and it has a lot to recommend. It might be inconclusive but the worries are real. It is certainly a precursor to subsequent criticisms of Frege, culminating in Davidson's demand for "semantic innocence." Although we share Russell's concerns, we do not share his conclusion to completely dispense of senses. The fact of the matter is that sense and mode of presentation are very powerful notions. Thus, what follows from this argument, we believe, is that the sense and reference *must not be separated* and sense should not be given primacy over reference. Insistence on there being entities which *determine* references leads to inconsistency, as Russell and his sympathizers think. Thus, a more general notion of sense which is not over and above reference is needed. Such demoted senses serve two important purposes: 1] addressing Frege's puzzles of informativeness of the identity and of substitution failures in opaque contexts; 2] avoiding inconsistencies, as demonstrated in Russell's argument. This also has an ontological advantage: sense and reference need not occupy different realms. The Fregean should not be offended by our proposal. For these "neo-senses" are still understood to be inter-subjective and non-psychological.

2 Sense-Reference Formalization

How to carry out the recommendations from §1? Before presenting our proposal, we briefly explore how the Fregean sense is formalized in the literature. The most widely accepted approach is due to Carnap. Realizing the shortcoming of sets (among other considerations), Carnap renders Frege's Sense as functions from possible worlds (PWs) to extensions—called *intensions*. But, since functions are set-theoretically special kind of relations, this approach in effect can be characterized as extensionalizing intensional natural language (NL) constructs. Although the introduction of PWs has led to impressive developments in semantics, PW-semantics has obvious drawbacks, e.g.,: 1) the ontology had to make room for PWs; 2) propositions are coarsely-individuated—a major disadvantage to AI, as cognitive agents' representation of states of affairs is at issue, not the mere truth or falsity thereof—for metaphysical, psycholinguistic, semantic or computational/cognitive concerns regarding PWs, see, respectively, (McGinn,2000:69), (Partee,1979), (Chierchia & Turnner,1988) and (Hirst,1988) and (Woods,1985).

In contrast to the Carnap-inspired approach just outlined, where some add-on is necessary to make NL expressions fit the rigid molds of classical sets, our approach is to generalize the notion of set to achieve a better sign-denotatum fit, thus, eliminating any appeal for extraneous notions.

This generalization is of interest by itself. For conventional sets have problems of their own. For instance, although set theory stresses the notion of aggregate—a permissive condition to what is to be a whole—it is restrictive on what can be admitted as a member, as only concepts with definite boundaries are suitable for set treatment. This is a clear bias toward mathematical and scientific concepts, while the majority—those which belong to the pre-theoretic sphere—are left out. This outlook seems to be in agreement with the view that only well-defined concepts are worthy of serious study.

However, set shortcomings are not limited to the pre-theoretic discourse. It is well known that sets cannot satisfactorily accommodate the most central notion in philosophy and logic: predication. This is noted by many scholars, e.g., (Cocchiarella, 1988). Also, it is inconceivable, according to the Cantorian conception of set, to talk about one object being a member in the same set in two distinct capacities or an object being partially a member of a set. Such talk runs afoul of the view that: a) sets are extensional objects first and foremost; b) set membership is a yes-no affair. These pose no problem to the development of mathematics, as mathematical objects are abstract and are well-defined and if this were not the case mathematicians and logicians would strive to make them so. Consequently, these objects have their properties absolutely and eternally. For instance, it would be an anomalous to describe a number as being "former even" or "potentially prime". This leads to the ideal situation where the meaning of a logico-mathematical term is its referent and the meaning of a predicate is the set (or class) of objects which fall under the predicate. Scholars who have qualms about abstracts/universals welcome this use of sets. It is seen to provide a better solution to the problem of properties than the postulation of universals. For, in such a view, talk about properties is reduced to talk about sets, which are well understood and have well-defined identity criterion—a fitting solution that goes along Quine's maxim of *no entity without identity*. But this extreme view alienates other fields such as AI and Semantics where mental representation, commonsense reasoning and other aspects of meaning are at issue.

3 Generalized Set: notion and notation

We conceive of an urelement as consisting of a "core" and a "type"; our choice of the term "type" is meant to be theory-free but in the context of NL analysis we believe it is better cast as Fregean Mode of Presentation, as suggested in §1. The core can exhibit many types. The type can be thought of as the member's public interface. It might be helpful to draw an analogy: an urelement can be thought of as a rotary rolodex. At some particular time a rolodex can be in one of two states: open (and presenting some contact information) or closed. In both cases it is still the same rolodex. Similarly, a member can participate in some type—corresponding to an open rolodex or it can participate in the generic type—analogous to that of a closed rolodex. In both cases, it is still the very same member.

The urelement, in addition to being part of an aggregate, can also participate in the type of the containing set. Although set members can be of the same type as the set, their types are distinct. What makes them of the same type is their participation in the type, i.e., that of the containing set. Our position has undeniable affinity to that of the Trope Theorist, (Loux 2006:71), where attributes are considered *particulars*—as opposed to *universals*—and a term, which the realist takes to stand for a universal, names a set of resembling tropes. However, while the trope theorist speaks of set—without qualification—we speak of *typed* set.

The notation we adopt is simply that of conventional set. In particular, we adopt the conventional set notation of braces to represent sets. Sets' type are designated by capital Greek, while members' by small Greek letters or a letter juxtaposed by a sequence of primes. Following this convention, a typed set may have the following form:

$$\{(b:\beta),(c:\gamma)\}:\Psi$$

Here, the set has two members, of type β and γ. The set is of type Ψ.

3.1 Formation Rules

Let τ be the set of basic types. Then, T is the smallest set such that:

i) $\tau \subseteq T$.

ii) if X and Y $\in$ T then (X op Y) $\in$ T.

op stands for either $\oplus$ or **:**, the type construction operators. They are used along the union and intersection operations.

The set of basic types, τ, also contains three special basic types: the top, the bottom and the hash. The top type, $\top$, is the most general type. It can be thought of as the disjunction of all types. Its polar opposite the bottom type, $\bot$, or the absurd type, is the conjunction of all types. The empty set, then, can uniquely be identified as having the bottom type, that is, $\varnothing:\bot$. It follows from this that there are many other empty sets which belong to different types or a combination thereof. The hash, #, is the vacuous type. It can be thought of as a place holder, see examples in §4.

The special case where members as all as the containing set are of the top type corresponds to the ZF set.

3.2 Operations on Sets

(Objects identity) Let ≡ stands for identity between objects. For any two set members x=*e1:t1* and y=*e2*:

$$x \equiv y \Leftrightarrow e1=e2 \wedge [(t1=t2)]$$

(**Type Casting/Shifting**) Typed set introduces a type-casting operation, which has no equivalent in conventional set. It has the following form:

$$(X)(e:y)$$

After this operation is carried out, the object **e** exhibits the type specified by the type **X**, i,e., the operation returns (e:x).

(**Subset**) for any sets X and Y of types T1 and T2, respectively:

$X \subseteq Y \Leftrightarrow \forall e\, [(e \in X \Rightarrow e \in Y)] \wedge (T1 = T2)$

(**Set equality**) For any sets X and Y of types, respectively, T1 and T2 :

$X = Y \Leftrightarrow [(X \subseteq Y \wedge Y \subseteq X) \wedge (T1 = T2)]$

(Union) For any sets X and Y of type T1 and T2, respectively:

$X \cup Y = \{x | x \in X \vee x \in Y\} : (T1 \oplus T2)$

(**Intersection**) For any sets X and Y of type T1 and T2, respectively:

a) $X \cap Y = \{x |\ x \in X \wedge x \in Y\} : (T1 : T2)$

b) $[(x \in X \wedge y \in Y) \wedge (x = (e:\top) \vee y = (e:\top))] \Rightarrow [(e:\top) \in (X \cap Y) = Z : (\top \oplus T1)]$, for any e.

(**Properties of the operators : and** $\oplus$). For any distinct types $T1 \neq \#$ and $T2 \neq \#$, where $\top$ and $\bot$ are the top and bottom types, the following hold:

a) $[T1 \oplus T2] = [T2 \oplus T1]$

b) $[T1 \oplus T1] = T1$

c) $[\top \oplus T1] = \top$

d) $[\bot \oplus T1] = T1$

e) $[T1 : T2] \neq [T2 : T1]$, where $T1 \neq \bot$ and $T2 \neq \bot$.

f) $[T1 : T1] = T1$

g) $[T1 : \bot] = \bot$.

h) $[\top : T1] = [T1 : \top] = T1$.

The identities k-m hold for the vacuous type, #:

k) $[\# : T1] = [T1 : \#] = T1$ l) $[\# \oplus T1] = [T1 \oplus \#] = T1$ m) For any x, $(x : \#) = (x : T1)$

3.3 Predication and Graded Membership

One of the motivations for typed sets is to better accommodate predication in areas where the classical set seems to be too limiting and to permit what we conceive of as an aggregate of disparate objects whose admission to the aggregate need not be definite. This involves allowing members of different membership levels. This aligns with our belief that predication is a special case of set membership: members of a non-empty denotation of predicate *both* belong to the aggregate and participate in the same type of the predicate. Example: let T be the denotation of a predicate term, say *F*, with members as follows:

$$T = \{(o:\alpha),(e:\alpha'),(b:\omega),(r:\omega'),(c:\gamma),(h:\pi),(t:\top)\}:\Omega$$

Although T has seven members, *F* can only be predicated of three members: *b*, *r*, and *t*. This is because the first two participate in the type Ω and *t* is member in some absolute sense. The members *o*, *e*, *c* and *h* belong to the aggregate but *F* cannot be predicated of any of them. This is indicated by neither of them having an *Ω* type, i.e., those syntactically formed by ω or ω followed by prime(s). Notionally, the significance of this distinction is to capture graded membership: some objects fall within the concept completely while others fall only partially. An example of the latter is being an observer in a committee or a group; or, describing someone as allegedly being a member of a gang. Also, it is possible that a member is participating in the aggregate in different capacity.

3.4 Quantification

Classical sets fit well with classical logic in terms of types assumed. Since a term in these systems is logical *par excellence* only the referent of the term plays a semantic role and nothing else beside the referent. So, the universal quantifier is rephrased for *anything* or *everything* and similarly the existential goes by the colloquial *at least something* is such and such. Here, the type is implicitly the top type, '⊤'. With typed sets in mind, we are in agreement with the classical theorist: the domain contains definite things. But we differ in insisting that members of a set must not necessarily be of top type. For the result of isolating part of the domain must be reflected in the resulting set. That is, the resulting set consists of objects from the domain as well as the way they are perceived. Here we live up to Cantor's words of "a collection into a whole of distinct objects of *our* intuition or thought[*]." Thus, in typed sets, although we speak of type as part of the member, it is meaningless to speak of typed objects of the domain of quantification. For there are indefinite ways the objects of the domain can be referred to and, as we have seen, typed sets may contain objects of different types. Consequently, the formulae of our language must specify the type under quantification. For instance, the universal quantifier must have the following form:

$$[(\forall x)\ P(x) \Rightarrow R(x)]_t$$

Here, the subscript indicates the type under quantification and the square brackets delineate the scope of the type *t*—thus, *t* (by default) is the top type, ⊤.

Thus, we have two kind scopes: that of quantification and that of type. These scopes may not necessarily coincide. For instance the discourse "*A cat is on the mat. It is black.*" can be rendered, assuming the definite description is a referring term:

$$\exists x[\mathrm{cat}(x) \wedge \mathrm{on}(x,\mathrm{the\_mat}) \wedge \mathrm{black}(x')]$$

Here, the anaphor is not c-commanded by the antecedent. So it cannot be bound by the quantifier. However, a co-variance between the antecedence and the anaphor is evident and needs to be accounted for: this is the task of type-binding, indicated by the prime in the argument of the predicate **black**. For type-binding needs not adhere to the c-command restriction. What matters here is the interpretation: it states that *x'* refers to a type (or mode of presentation) which can be used to determine the referent, denoted by *x*, for the predicate. This is possible since *x* and *x'* share the same unique type.

4 An Example

In this section we show how the meaning of some NL constructs can be represented and computed using typed sets. We assume the Direct Theory of Reference, *à la* Kripke, regarding proper names as well as our newly Frege-Russell proposition. The latter is thought as a pair of the form **<object, S>**, which is true if, and only if, the object named is a member of the typed set S.

Although we cannot go into much detain, we assume a model containing a set of the objects of the domain and a set of types—for of every construct of the object language with the exception of connectors and proper names. Computation proceeds compositionally following the rules of §3 and function application.

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[*] Emphasis is ours.

Modification

Adjective-noun combinations are believed to form a semantically-motivated 3-level-hierarchy, (Kamp and Partee, 1995). Conceived as such, these compounds seem to challenge the syntax-semantics doctrine. The accepted uniform treatment rests on PW semantics.

Consider, a) *That is a red car* and b) *Alice is a good pianist*. If the denotations of both *red* and *car* are taken as sets, then (a) is true if, and only if, the referent of *that* belongs to the intersection of the two sets. But (b) has quite different truth-conditions. The adjective *good* in (b) is said to have two possible readings: reference-modifying and referent-modifying. The former pertains to Alice herself, that is, her character. The latter, pertains to her piano performance. The trouble starts when an at-face-value ZF set-theoretic treatment is given to *good* similar to that of *red* in (a). For Alice may happen to be a teacher too. This would wrongly lead to the conclusion that Alice is a good teacher which doesn't follow from (b) and the sentence *Alice is a teacher*. With the distinction between set-membership and predication in mind, type-endowed sets can correctly and uniformly encode the truth-conditions of the adjective hierarchy. To demonstrate, consider (b), again, and c) *Alice is a teacher*. In our account, common nouns and *all* adjectives denote (typed) sets. So, the facts from (b) and (c) are represented as follows, where the symbol '‖ ‖' designates the denotation function:

$$||Alice||=(e:\#) \text{ or just } e \qquad ||good||=\{(e:\gamma),(e:\top),..\}:A$$
$$||teacher||=\{(e:\beta),..\}:B \qquad ||pianist||=\{(e:\gamma),..\}:\Gamma$$

It should be obvious that the inference *Alice is a good teacher* cannot be reached, as the member (e:β) does not belong to the intersection of the denotations of *teacher* and *good*. It should be noted that Alice is a member of the set ‖good‖ in two different capacities. It follows by rule Intersection-(b) of §3.2 that the assertion *Alice is a teacher who is good* holds. This is because (e:⊤) is in the intersection of *teacher* and *good* and the resultant set is of type ⊤.

Adjectives of the third level are characterized by having referents that neither belonging to the denotations of the intersection of the adjective and the common noun nor to a subset of the denotation of the common noun, (d) and (e) are examples:
d) *That is a fake gun* and e) *Lefty is an alleged murderer*. With *fake* and *gun* are assigned the types A and B, respectively, (d) and (e) can be rendered as follows:

$$||that||=g \qquad ||fake||=\{(g:\alpha),..\}:A \qquad ||gun||=\{(g:\alpha),..\}:B$$

We immediately see that we can predicate fake of g but not gun. Thus, we can say about the object g that it is fake—for it is in ‖fake‖ and has the same type as the set—but we cannot describe it as a gun. (d) can be rendered as follows:

$$||Lefty||= l \qquad ||alleged||= \{(l:\psi),..\}:\Omega$$
$$||murderer||=\{(l:\omega),..\}:\Psi$$

Here, Lefty can neither be predicated of alleged nor murderer. But the statement *Lefty is alleged murder* can be asserted as follows, where F is a function from types and sets to sets and T is a function from sets to types:

$$||Lefty||\in[F(T(||murderer||),||alleged||) \cap ||murderer||]$$

The extra work here is needed since, as things stand, the intersection of the denotation of *alleged* and *murderer* is empty. To get the correct reading, we need to do two things: to extract members of type ‖murderer‖ from the set ‖alleged‖ and to form a new set with members of type ‖alleged‖. This is what the function F is purported to do. Then intersection can be carried out. The whole expression, then, can be evaluated to True if, and only if, ||Lefty|| is in the intersection.

Noun-noun combinations can be handled in a similar way. Expressions such as child murderer are ambiguous—describing a murderer who happens to be a child or a child who is a murderer. *Sparky is a child murder* can be rendered as follows:

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||Sparky||=s                   ||child||={(s:σ),(s:ψ),..}:Σ
||notorious||={(s:ψ),..}:X     ||murderer||={(s:σ),(s:ψ),..}:Ψ
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If it happens that Sparky is a notorious murder then the assertion Sparky is notorious child murderer is derivable since (s:ψ) belongs to the intersection of the three sets. It should be noted that given membership-predication distinction the assertion Sparky is notorious is semantically ill-formed as does Lefty is alleged. This is reflected in the representation by having both Spark and Lefty as members but not type-participants in the respective sets, making predication unattainable.

5 Concluding Remarks

It might be objected that typed sets are extraneous. For the set theory is a framework for all of mathematics and pairs, or any other structure, would do just fine?

Prima facie it seems harmless to render a typed set member as a pair, say <a, b>. However, according to the most commonly accepted definition of a pair, this is just a short for the set {a, {a, b}}. But this would treat types as having an equal ontological status to and as being independent of their bearers. Since a particular's redness cannot be isolated from the bearing particular, similarly a type cannot be separated from its bearer. When we say a member may bear many types, we do not mean an object can acquire and shed its types at will. For types are intrinsically dependent on their bearers.

This, however, doesn't mean that typed sets cannot be represented as pair. They can, as long as the conceptual framework is taken into consideration. Calculus without the conceptual leap of the derivative would remain the old algebra. But once the underlying notion of the derivative was fully appreciated, the algebraic machinery was called upon to carry it out. Similarly, pairs, or any other mathematical structure, can be used as long as the underlying notion of typed sets is honored.

It might, also, be objected that types violate the ontological parsimony. But the classical sets implicitly assume them. This is true not only of Russell's Type Theory and theories which assume urelements. Types are, also, present in ZF-axiomatic set theory. It might be argued that in the latter types exist only pre-theoretically. However, our way of utilizing types is a natural way to extend the notion of set where not only the aggregate's size is of interest to us but also the fabric of its composition—*ad hoc* vs natural totalities.

Finally typed sets seem to have the potential for a viable treatment of anaphora, as alluded to in §3.4, and going beyond sentence/clause boundaries.

6 Conclusion

While the literature on structured propositions is divided between the Fregean and the Russellian, we sought a unified approach whereby not only particular individuals are constituents but also watered-down Senses, thus taking advantage of both appealing conceptions. We presented typed sets as a vehicle to execute this program. We argued that typed sets are of interest on their own right, as they cover conceptual landscape neither fuzzy sets nor classical sets do. We showed how predication, aggregate and quantification can be viewed from a typed set perspective. Finally, we demonstrated how non-extensional constructs can compositionally be accommodated using typed sets, without resorting to additional notions and requiring no more resources than those available to first-order sets.

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